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MODEL THEORY AND SPECTRA

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Given an extension $\mathcal{T}$ of the theory of commutative rings with 1 admitting elimination of quantifiers, we set up a spectrum functor $\mathcal{T}$-Spec from the category of commutative rings with 1 into the category of topological spaces. $\mathcal{T}$-Spec A consists of equivalence classes of homomorphisms of A into models of $\mathcal{T}$. 'Closed' formulae - defined as those equivalent to a positive quantifier-free formulae - are used to define a topology on $\mathcal{T}$-Spec A making it into a spectral space. Additional properties, such as the existence of 'enough prime models' and 'closed projections' are defined and discussed. If there are enough prime models, we define affine space in a way familiar to algebraic geometers. Affine space enjoys an Artin-Lang property. Finally, we define sheaves of definable sections and continuous definable sections which turn out to be classical sheaves in the classical examples of the Zariski and real spectra. Notions such as continuous, closed, bounded, and having stalks which are local rings are defined in a purely model-theoretic way, and we are able to prove some rather general continuity results about $\mathcal{T}$-Spec as a functor into the category of locally ringed spaces. We pay considerable attention to a new example - the *real valuation spectrum* - based upon Cherlin and Dickmann's work on the theory of real closed valuation rings. Our set-up allows us to relate continuous definable sections over the real spectrum to definable sections over the real valuation spectrum.

0. Introduction

The Zariski spectrum is a functor from the category of commutative rings with 1 into the category of locally ringed spaces. It plays a fundamental role in algebraic geometry, especially over algebraically closed fields. More recently, real (or semialgebraic) geometers have used the real spectrum and its sheaves of abstract functions as fundamental object, see [1,5,11]. The p-adic spectrum, for use in p-adic geometry, is the subject of a recent monograph by Bröcker and Schinke [2], Stengle has started work on the real differential spectrum [13], and a version of the real valuation spectrum appears in a paper by Schwartz [12].

The purpose of this paper is to construct a model-theoretic framework unifying these and other spectrum functors which (might) have algebro-geometric significance. The basic idea is that points of the spectrum of a ring are equivalence classes

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of homomorphisms into models of a fixed theory. The theories of interest are theories of rings with additional structure and which admit quantifier elimination in a 'natural' language.

This basic setup is described in Section 1. In Section 2 we use positive quantifier-free formulae to define closed sets, thereby generating a topology mimicking the Zariski topology and the natural topology on the real spectrum. Using Stone spaces we can quickly derive basic facts about our topologies.

In Section 3 we address the problem of which theories actually generate nice geometric objects. In order to relate quantifier elimination to our topology, we are led to consider substructure homomorphism extension problems. This relates the algebra of models of our theory to the spectra they produce. Our inspiration comes from Van den Dries' paper [14].

As a test for our topological theory, in Section 4 we consider the theory of real closed valuation rings, studied by Cherlin and Dickmann in [4]. Here it turns out that various geometric considerations about our spectrum lead to variations on the language. Since the choice of a language for a given theory is somewhat arbitrary, we view our machinery as providing guidelines for this choice. The spectrum functor constructed (the real valuation spectrum) is new, so we go into some detail as to its geometric nature.

In Section 5 we define affine space and study the spectra of 'affine coordinate rings' over a model R of a given theory. In general, affine space is considered as fibred over the spectrum of R. Again, we go into some detail concerning the real valuation spectrum.

In Sections 6 and 7 we introduce our structure sheaves as sheaves of 'abstract functions'. These reduce to normal (fibred) functions on affine space. The functions introduced in Section 6 are simply definable functions, but they do not relate to the more delicate topologies on our spectra. In Section 7 we address the question of defining continuous functions and are once again led to a homomorphism extension property needed to prove that our functions form a ring. We show that a definable function on the real specrum of any ring is an abstract semialgebraic function if and only if its graph is closed and it extends to a continuous function on the real valuation spectrum.

The sheaves we introduce have the property that sections come from sections over finitely generated subrings. Finally, in Section 8 we codify the notion of a definable sheaf of functions, all of which have this property. Questions about finitely many sections of such sheaves can be reduced to questions about functions on closed constructible subsets of affine space, and this gives us a generalized Tarski principle. This has been used, for example, in [9].

We hasten to mention that the model theory used in this paper is quite elementary and should be understandable to algebraists in general and to the author in particular. We also mention that we only discuss the two classical examples (the Zariski and real spectrum) and the real valuation spectrum. A good test of our machinery is how significant other spectra will be in the future.

1. The functor $\mathcal{T}$-Spec : CRO → TOP

Throughout this paper we let CRO denote the category of commutative rings with 1 and homomorphisms preserving 1. In this section we start with a theory whose models are rings and use it to define a basic functor from CRO into the category TOP of topological spaces with continuous maps. Although this can be done for general theories, we will make some very strong assumptions which we will need later anyway.

Let $\mathcal{L}$ be a first order language containing at least 0 and 1 as constants and $+$ and $\times$ as functions. Let $\mathcal{T}$ be a theory in $\mathcal{L}$ satisfying

Property 1.1. (i) $\mathcal{T}$ contains the axioms for commutative rings with 1.
(ii) $\mathcal{T}$ admits elimination of quantifiers.

Let A be a commutative ring with 1 and let $\mathcal{L}_A$ be the language obtained by adding constants from A. Let Δ^+A denote the positive diagram of A in the language of rings (see [3]) and let $\mathcal{T}_A = \mathcal{T} \cup \Delta^+A$. Then a model of $\mathcal{T}_A$ is simply a model $\mathcal{M}$ of $\mathcal{T}$ together with a homomorphism $\alpha : A \to \mathcal{M}$. If $a \in A$, then $\alpha(a)$ is the realization of the constant a in $\mathcal{M}$.

Definition 1.2. The set of models of $\mathcal{T}_A$ modulo elementary equivalence is called the *$\mathcal{T}$-spectrum of A* and is denoted by $\mathcal{T}$-spec A.

Since $\mathcal{T}$ admits quantifier elimination, two models of $\mathcal{T}_A$ are elementarily equivalent if and only if they satisfy the same quantifier-free (abbreviated q.f.) sentences in $\mathcal{L}_A$. Using this, it is easy to check that, for example, if $\mathcal{T}$ is the theory of algebraically closed fields, then $\mathcal{T}$-spec A is the usual Zariski spectrum of A, and that if $\mathcal{T}$ is the theory of real closed fields in the language $\mathcal{L}\langle +, \times, 0, 1, \leq \rangle$, then $\mathcal{T}$-spec A is the real spectrum of A.

Let $Q_{\mathcal{T}}(A)$ denote the boolean algebra of q.f. sentences in $\mathcal{L}_A$ modulo $\mathcal{T}_A$-equivalence, so $\theta(\bar{a}) \sim \psi(\bar{b})$ iff $\mathcal{T}_A \vdash \theta(\bar{a}) \Leftrightarrow \psi(\bar{b})$. Then $\mathcal{T}$-spec A is the Stone space of ultrafilters of $Q_{\mathcal{T}}(A)$ (see [3]). This has a natural topology in which basic open sets consist of those ultrafilters containing a given sentence. We translate this to $\mathcal{T}$-spec A as follows:

Definition 1.3. If $\theta(\bar{a})$ is a sentence in $\mathcal{L}_A$, we define

$$\mathcal{X}(\theta) = \{[\mathcal{M}] \in \mathcal{T}\text{-spec } A \mid \mathcal{M} \models \theta\}.$$

Here $[\mathcal{M}]$ denotes the equivalence class of the model $\mathcal{M}$. Sets of this form are called *constructible*. The constructible sets form a boolean algebra denoted by $\mathcal{C}(\mathcal{T}\text{-spec } A)$. This also serves as a (restricted) topology, the *constructible topology*.

Remark. The reader might object to the word "topology", since we do not demand closure under arbitrary union. A topology closed under finite union is certainly a

Grothendieck topology, but to avoid this level of generality we prefer to use the term *restricted topology* introduced by Delfs and Knebusch in [7].

Every CRO-morphism $f: A \to B$ gives a map $f^*: \mathscr{T}\text{-spec}\, B \to \mathscr{T}\text{-spec}\, A$ sending the equivalence class $[\alpha]$ of a homomorphism $\alpha: B \to \mathscr{M}$ to the equivalence class $[\alpha \circ f]$. We have

Proposition 1.4. *$\mathscr{T}$-spec is a contravariant functor* CRO $\rightsquigarrow$ TOP *whose image lies in the subcategory of compact Hausdorff spaces.*

Proof. We clearly have $(f \circ g)^* = g^* \circ f^*$, and Stone spaces are compact Hausdorff spaces, so we only need to show that f^* is continuous, i.e. that the inverse image of a constructible set is constructible. Let $\theta(a_1, \dots, a_n)$ be a sentence in $\mathscr{L}_A$. Then a homomorphism β from B into a model $\mathscr{N}$ of $\mathscr{T}$ represents a class in $f^{*-1}\mathscr{X}(\theta)$ if and only if $\mathscr{N} \models \theta(\beta(f(a_1)), \dots, \beta(f(a_n)))$, so $f^{*-1}(\mathscr{X}(\theta(a_1, \dots, a_n))) = \mathscr{X}(\theta(f(a_1), \dots, f(a_n)))$. $\square$

In general, images of constructible sets are not constructible. However we do have:

Proposition 1.5. *If $f: A \to B$ is a* CRO*-morphism giving B as finitely presented over A, then f^* is an open map in the constructible topologies.*

Proof. We can factor f as $\pi \circ \iota$ where $\iota: A \to A[T_1, \dots, T_n]/\mathscr{I}$ is the obvious inclusion, $\pi: A[T_1, \dots, T_n] \to A[T_1, \dots, T_n]/\mathscr{I}$ is the conical projection, and $\mathscr{I} = \langle g_1, \dots, g_m \rangle$ is finitely generated. A homomorphism $\alpha: A[T_1, \dots, T_n] \to \mathscr{M}$ into any ring extends to B if and only if there are $m_1, \dots, m_n \in \mathscr{M}$ such that all $g_i(m_1, \dots, m_n) = 0$. Choose $G_i \in A[T_1, \dots, T_n]$ such that $\pi(G_i) = g_i$. If $\theta(b_1, \dots, b_k)$ is a sentence in $\mathscr{L}_B$, we may choose polynomials $B_1, \dots, B_k \in A[T_1, \dots, T_n]$ with $B_j \equiv b_j \pmod{\mathscr{I}}$. Then $f^*\mathscr{X}(\theta(b_1, \dots, b_k))$ is defined by "$\exists m_1, \dots, m_n\ (G_1(\bar{m}) = \dots = G_m(\bar{m}) = 0 \wedge \theta(B_1(\bar{m}), \dots, B_k(\bar{m})))$". $\square$

Proposition 1.6. *Let $\{A_i\}_{i \in I}$ and $\{f_{ij}: A_i \to A_j\}_{i \le j}$ be a filtered inductive system in* CRO*. Let $\iota_i: A_i \to \varinjlim A_i$ denote the canonical inclusion. Then $\varprojlim \iota_i^*$: $\mathscr{T}\text{-spec}(\varinjlim A_i) \to \varprojlim \mathscr{T}\text{-spec}\, A_i$ is a homeomorphism.*

Proof. There is an equivalence of categories between Stone spaces of boolean algebras and spectra of their associated boolean rings. Using this, the result follows from the same continuity result for the Zariski spectrum once we know that the assignment $A \mapsto Q_{\mathscr{T}}(A)$ is functorial and preserves direct limits. This is straightforward to check (or see the proof of Proposition 6.4 below). $\square$

2. The spectral topology

The naturally occurring topologies on the real and Zariski spectra are weaker than the constructible topologies. In order to generalize these, we use *positive* q.f. formulae – i.e. q.f. formulae with no negated atoms – to define closed sets. In other words, we define

Definition 2.1. A formula $\theta(T_1,\dots,T_n)$ in free variables $T_1,\dots,T_n$ is called ($\mathcal{T}$-)*closed* if there is a positive q.f. formula $\psi(T_1,\dots,T_n)$ such that $\mathcal{T}\vdash\psi\Leftrightarrow\theta$. A formula θ is *open* if $\neg\theta$ is closed. A constructible set $X\subset\mathcal{T}\text{-spec}\,A$ is called closed (open) if $X=\mathcal{X}(\theta(a_1,\dots,a_n))$ for some closed (open) formula $\theta(T_1,\dots,T_n)$. The lattice of closed constructible subsets of $\mathcal{T}$-spec A is denoted by $\bar{\mathscr{C}}(\mathcal{T}\text{-spec}\,A)$, and the lattice of open constructible sets is denoted by $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$. $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ is a restricted topology and generates a standard topology (with infinite unions) for $\mathcal{T}$-spec A known as the *spectral topology.*

The reader should note that this definition involves both $\mathcal{T}$ and the language $\mathcal{L}$ in which $\mathcal{T}$ is expressed.

Recall that a topological space is a *spectral space* if it has a basis of quasi-compact open sets, is T_1, and if every irreducible closed set is the closure of a single point (contains a generic point). These spaces were introduced by Hochster [8]. He showed they are precisely the images of the Zariski spectrum functor Spec : CRO $\rightsquigarrow$ TOP. To show that $\mathcal{T}$-spec A is spectral, we use

Lemma 2.2. *Let $\mathcal{F}$ be an ultrafilter in $\mathscr{C}(\mathcal{T}\text{-spec}\,A)$ and let*

$$\mathcal{F}^\circ=\{U\in\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)\mid U\in\mathcal{F}\}.$$

Then $\mathcal{F}^\circ$ is a prime filter in $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$. Conversely, suppose that $\mathcal{G}\subset\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ is a prime filter. Let $\tilde{\mathcal{G}}=\{X\in\mathscr{C}(\mathcal{T}\text{-spec}\,A)\mid X\cap U\neq\emptyset$ for all $U\in\mathcal{G}\}$. Then $\tilde{\mathcal{G}}$ is the unique ultrafilter with $\tilde{\mathcal{G}}^\circ=\mathcal{G}$.

Proof. Standard, since $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ generates $\mathscr{C}(\mathcal{T}\text{-spec}\,A)$. □

Corollary 2.3. *$\mathcal{T}$-spec A is a spectral space in the spectral topology.*

Proof. We view points of $\mathcal{T}$-spec A as prime filters in $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$. If $\alpha,\beta\in\mathcal{T}\text{-spec}\,A$ and $\alpha\neq\beta$, then by definition there is a $U\in\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ with $U\in\alpha$, $U\notin\beta$, or vice versa. Since $U\in\alpha$ (as a set contained in a filter) if and only if $\alpha\in U$ (as a point contained in a subset of $\mathcal{T}$-spec A), we see that $\mathcal{T}$-spec A is T_1. All sets in $\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ are quasi-compact in the stronger constructible topology, so all we have to verify is the existence of generic points.

Suppose $K\subset\mathcal{T}\text{-spec}\,A$ is closed and irreducible and that we are given $U_1,\dots,U_k\in\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)$ with $U_i\cap K\neq\emptyset$ for each i. Then $K\cap U_1\cap\cdots\cap U_k\neq\emptyset$. Thus $\{U\in\mathscr{C}^\circ(\mathcal{T}\text{-spec}\,A)\mid K\cap U\neq\emptyset\}$ forms a filter α which is clearly prime. Certainly $\alpha\in\bar{K}$, so $\bar{\alpha}\subset\bar{K}=K$. But if $\beta\in K$ and U is an open constructible set containing β, then $K\cap U\neq\emptyset$, so $\alpha\in U$. Thus $K\subset\bar{\alpha}$, and $K=\bar{\alpha}$. □

Note that if $f:A\to B$ is a CRO-morphism, then the functorial map $f^*:\mathcal{T}\text{-spec}\,B\to\mathcal{T}\text{-spec}\,A$ is continuous in the spectral topologies. Also, Proposition 1.6 still holds

- This can be seen directly as in the proof of Proposition 6.4 or by using a different topology on $Q_{\mathscr{T}}(A)$ and checking that the homeomorphisms involved are also spectral homeomorphisms.

We now state and a model-theoretic lemma from [14]. The interpretation (given below) in terms of specializations derives from the same work.

Lemma 2.4. *Let $\theta(T_1,\dots,T_n)\in\mathscr{L}$. Then θ is $\mathscr{T}$-closed if and only if it satisfies the following criterion: Given any model $\mathscr{M}\models\mathscr{T}$, any substructure $S\subset\mathscr{M}$, any $s_1,\dots,s_n\in S$, and any $\mathscr{L}$-homomorphism $f:S\to\mathscr{N}$ into a model $\mathscr{N}\models\mathscr{T}$, then $\mathscr{M}\models\theta(s_1,\dots,s_n)\Rightarrow\mathscr{N}\models\theta(f(s_1),\dots,f(s_n))$.* $\square$

A very useful concept for understanding the structure of spectral spaces is that of *specialization*. If S is a spectral space and $\alpha,\beta\in S$, then we say that β specializes α, and write $\alpha\to\beta$, if $\beta\in\bar{\alpha}$.

If $\alpha\in\mathscr{T}\text{-spec}\,A$ is represented by a homomorphism into $\mathscr{M}$ and $\theta(a_1,\dots,a_n)\in\mathscr{L}_A$, we often write $\theta(\alpha(a_1),\dots,\alpha(a_n))$ to mean that $\mathscr{M}\models\theta(\alpha(a_1),\dots,\alpha(a_n))$. So "$\alpha(x)>\alpha(y)$" means, where appropriate, that x has a larger image than y under α. By the definition of the spectral topology, $\alpha\to\beta$ if and only if $\theta(\alpha(a_1),\dots,\alpha(a_n))\Rightarrow\theta(\beta(a_1),\dots,\beta(a_n))$ for all positive atomic formulae θ.

This allows us to interpret the homomorphism situation of Lemma 2.4 in terms of specialization. If $\mathscr{M}$ and $\mathscr{N}$ are models of $\mathscr{T}$, and S is a substructure of $\mathscr{M}$, then S is itself a ring and $\mathscr{M}$ represents a point $\alpha\in\mathscr{T}\text{-spec}\,S$. The existence of a ring homomorphism $f:S\to\mathscr{N}$ means $\mathscr{N}$ represents a point $\beta\in\mathscr{T}\text{-spec}\,S$, and f is also an $\mathscr{L}$-homomorphism if and only if $\alpha\to\beta$.

Suppose we start with a commutative ring A with 1. If $\alpha,\beta\in\mathscr{T}\text{-spec}\,A$ are represented by homomorphisms into $\mathscr{M}_\alpha$ and $\mathscr{M}_\beta$, we can consider the substructure S_α of $\mathscr{M}_\alpha$ generated by $\alpha(A)$. $S_\alpha=\alpha(A)$ if there are no constants and functions in $\mathscr{L}$ other than those from the theory of rings. If $\alpha\to\beta$, sending $\alpha(a)$ to $\beta(a)$ induces a map S_α to S_β which is a well-defined $\mathscr{L}$-homomorphism from S_α into $\mathscr{M}_\beta$. Thus Lemma 2.4 implies

Corollary 2.5. *A constructible subset of $\mathscr{T}$-spec A is spectrally closed (open) if and only if it is closed under specialization (generalization).* $\square$

Applying this to functions, we obtain

Corollary 2.6. *Let $f:\mathscr{T}\text{-spec}\,A\to\mathscr{T}\text{-spec}\,B$ be a function which is continuous in the constructible topology. Then f is spectrally continuous if and only if $f(\alpha)\to f(\beta)$ whenever $\alpha\to\beta$. Let $f:\mathscr{T}\text{-spec}\,A\to\mathscr{T}\text{-spec}\,B$ be a function which is open in the constructible topology – i.e. which maps constructible sets to constructible sets. Then f is a spectrally open mapping if and only if given $\beta\in\mathscr{T}\text{-spec}\,A$ and $\gamma\in\mathscr{T}\text{-spec}\,B$ with $\gamma\to f(\beta)$ we can find an $\alpha\to\beta$ in $\mathscr{T}\text{-spec}\,A$ with $f(\alpha)=\gamma$.* $\square$

3. Additional structure

Both the assumption that $\mathscr{T}$ admits elimination of quantifiers and the definition of the spectral topology are highly language dependent. In order to have a good theory of 'affine space', we may need further restrictions on our theories. We concentrate on two such restrictions. The first concerns the property that the projection of an open set is open. It connects the topology with the quantifier elimination. We give some equivalent formulations.

Lemma 3.1. *Let $\mathscr{T}$ be a theory satisfying Property* 1.1. *Then the following two conditions are equivalent*:

(i) *If $\theta(T, U_1, \ldots, U_n)$ is a $\mathscr{T}$-open formula, then $\exists T\theta(T, u_1, \ldots, u_n)$ is $\mathscr{T}$-open.*

(ii) *For any commutative ring A with* 1 *the map $\pi : \mathscr{T}\text{-spec}\, A[T] \to \mathscr{T}\text{-spec}\, A$ induced by the inclusion $A \hookrightarrow A[T]$ is a spectrally open map.*

Furthermore, these are implied by the following condition:

(iii) *Given any models $\mathscr{M}$, $\mathscr{N}$ of $\mathscr{T}$ and any homomorphism $f : S \to \mathscr{N}$ of a substructure $S \subset \mathscr{M}$ into $\mathscr{N}$, there is a substructure $\tilde{S}$ with $S \subset \tilde{S} \subset \mathscr{M}$ and a homomorphism $\tilde{f} : \tilde{S} \to \mathscr{N}$ extending f such that for any $\mathscr{T}$-closed formula $\theta(T, U_1, \ldots, U_n)$ and $u_1, \ldots, u_n \in \tilde{f}(\tilde{S})$, $\mathscr{N} \vDash \forall T\theta(T, u_1, \ldots, u_n) \Leftrightarrow \tilde{f}(\tilde{S}) \vDash \forall T\theta(T, u_1, \ldots, u_n)$.*

Proof. The equivalence of (i) and (ii) follows from proposition 1.5, its proof, and Corollary 2.6. To see that (iii) implies (i), we apply Lemma 2.4 to the negated form of (i). So let $\theta(T, U_1, \ldots, U_n)$ be a closed formula and suppose $s_1, \ldots, s_n \in S$ and that $\mathscr{M} \vDash \forall T\theta(T, s_1, \ldots, s_n)$. We extend f to $\tilde{f} : \tilde{S} \to \mathscr{N}$ as in our hypotheses. Then $\tilde{S} \vDash \forall T\theta(T, s_1, \ldots, s_n)$, and since θ is closed, $\tilde{f}(\tilde{S}) \vDash \forall T\theta(T, \tilde{f}(s_1), \ldots, \tilde{f}(s_n))$. Hence $\mathscr{N} \vDash \forall T\theta(T, \tilde{f}(s_1), \ldots, \tilde{f}(s_n))$. $\square$

Remark 3.2. Since $\mathscr{T}$ admits quantifier elimination, the easiest way to show Lemma 3.1(iii) is to show that $\tilde{f}$ and $\tilde{S}$ can be chosen so that $\tilde{f}(\tilde{S}) \vDash \mathscr{T}$.

Definition 3.3. A theory $\mathscr{T}$ satisfying Property 1.1 and any of the conditions in Lemma 3.1 is said to have *open projections*. If $\mathscr{T}$ satisfies Lemma 3.1(i) with "open" replaced by "closed", then $\mathscr{T}$ has *closed projections*.

Finally, we introduce a condition which gives us prime models:

Definition 3.4. A theory $\mathscr{T}$ satisfying Property 1.1 is said *to have enough prime models* if, given any commutative ring A with 1 and any model $\mathscr{M}$ of $\mathscr{T}_A$, the complete theory of $\mathscr{M}$ in the language $\mathscr{L}_A$ has a 'prime model' – i.e. one which embeds in all others.

Notation. In this case we use k_α to denote a prime model associated with $\alpha \in \mathscr{T}\text{-spec}\, A$ and represent α by the homomorphism $\alpha : A \to k_\alpha$.

Of course k_α need not be definable over $\mathscr{T}_A$. In other words, there may be $\mathscr{L}$-automorphisms of k_α fixing $\alpha(A)$ as in the case of the algebraic closure of a non-algebraically closed field. We thus define:

Definition 3.5. A theory $\mathscr{T}$ with enough prime models is *rigid* if given any $A \in \mathsf{CRO}$ and any homomorphism $\varphi : A \to \mathscr{M}$ into a model of $\mathscr{T}$ representing a point $\alpha \in \mathscr{T}\text{-spec}\,A$, there is precisely one way to make the following diagram commute:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, dashed] & & \mathscr{M} \\ & k_\alpha \arrow[ur, "\alpha"'] & \end{tikzcd}$$

The two best-known examples of theories with open projections and enough prime models are:

Example 1 (the theory ACF of algebraically closed fields in the language $\mathscr{L}\langle 0, 1, +, \times \rangle$). For this theory, it is well known that Property 1.1 is satisfied. We check Lemma 3.1(iii) using Remark 3.2. Thus we are given an ACF L, a subring $S \subset L$ containing 1, and a homomorphism $f : S \to K$ into an ACF K. Let $(\tilde{S}, \tilde{f})$ be a maximal pair, where $\tilde{S}$ is a subring of L containing S and $\tilde{f} : \tilde{S} \to K$ extends f. By localizing at the kernel if need, we see that $\tilde{f}(\tilde{S})$ is a field $k \subset K$. If $x \in K$ is algebraic over k and p is its minimal polynomial, we may choose a polynomial $\pi \in \tilde{S}[T]$ so that applying $\tilde{f}$ to its coefficients gives p. We can then extend $\tilde{f}$ to $\tilde{S}[\xi]$ where ξ is a root of π in L. Thus $x \in k$, so k is algebraically closed.

If $f : A \to K$ is a homomorphism into an ACF representing the point $\alpha \in \operatorname{Spec} A$, then we can take k_α to be any algebraic closure of the quotient field $\mathrm{QF}(A/\mathrm{Ker}(f))$. In particular, ACF is not rigid.

Example 2 (the theory RCF of real closed fields in the language $\mathscr{L}\langle 0, 1, +, \times, \geq \rangle$). Again, Property 1.1 and the existence of enough prime models are well known or clear. The argument for (iii) goes exactly as in Example 1, once one realizes that a homomorphism in $\mathscr{L}$ is an order-preserving ring homomorphism. Thus one can localize at the convex kernel and extend to roots of odd degree polynomials and of polynomials $X^2 - a$ for $a \geq 0$.

The great advantage of RCF over ACF is that it is rigid.

Playing devil's advocate with RCF yields another example:

Example 3 (the theory RCFS of real closed fields with *strict inequality*, and *no equality*). Property (1.1)(ii) and the existence of enough prime models are immediate from the corresponding properties of RCF. All homomorphisms of totally ordered structures in this language are monomorphisms. The functors RCF-Spec (the usual real spectrum) and RCFS-Spec are the same except that open sets are now closed.

Thus RCFS has closed projections. According to a private communication from Schwarz, this can be applied to the abstract theory of 'weak semialgebraic spaces' (cf. [11]). We do not know at this point how well Schwarz' theory fits in with generalities presented here, nor do we pursue this theme.

4. Real closed valuation rings

In this section we discuss a fourth example of a theory satisfying Property 1.1, namely the theory of real closed valuation rings (RCVR). This theory has been extensively studied by Cherlin and Dickmann [4]. It is very close to the theory of real closed fields, and it appears to have some advantages, not all of which we discuss. We will only go into the topological (and later sheaf-theoretic) aspects far enough to illustrate the model theory and do some basic comparison with the real spectrum. These comparisons are meant for those familiar with the real spectrum.

Cherlin and Dickmann use a 'divides' predicate governed by the standard axiom

$$\mathrm{DIV}: \forall X, Y (X \mid Y \Leftrightarrow \exists Z (YZ = X)).$$

We begin with a result from [4]:

Proposition 4.1. *Let R be an ordered integral domain. Then the following are equivalent* (*rings contain* 1):

(a) *Every polynomial in $R[X]$ which changes sign on R has a root in R* (*R is a real closed ring*).

(b) *R is a convex subring of a real closed field.*

(c) *R satisfies the following three conditions*:

  (i) $0<a<b \Rightarrow b \mid a$ (*where* $\mid$ *is defined by* DIV).

  (ii) *Every positive element has a square root.*

  (iii) *Every monic odd degree polynomial over R has a root in R.* □

From (b) it follows that these conditions define specify real valuation rings in real closed fields, but they need not be proper. The theory RCVR of *real closed valuation rings* is the theory of ordered integral domains satisfying (c) but *which are not fields*. The main result from [4] is that RCVR admits quantifier elimination in the language $\mathscr{L} = \mathscr{L}\langle 0, 1, +, \times, \leq, \mid \rangle$.

Unlike in the case of real closed fields, it is perfectly possible to have a non-elementary containment $L \subset R$ of RCVR's. In fact, such a containment is elementary if and only if divisibility in L (as defined by DIV) agrees with that in R, see [4].

There are several choices of language for RCVR which depend upon what we want as spectrally closed sets. For example, the theory of RCVR's has closed projections in the language described above and open projections in the language obtained by considering '$<$' and '$\nmid$' as positive predicates. This may well be useful, but for this paper we will introduce another choice of predicate which allows us some control over non-units.

Let $\mathscr{F}$ be any real closed field and let R be the field of real algebraic Puiseux series over $\mathscr{F}$ in one variable, say T. Then R is real closed and T is positive and infinitesimal. *We define $\mathscr{O}(\mathscr{F})$ to be the valuation ring of Puiseux series with no negative powers of T.* The maximal ideal $\mathfrak{m}$ of $\mathscr{O}(\mathscr{F})$ consists of elements with zero constant term, and $\mathscr{O}(\mathscr{F})/\mathfrak{m} \cong \mathscr{F}$.

More generally, suppose A is any commutative ring with 1 and let $\alpha \in$ RCF-spec A be a point represented by a homomorphism $\alpha : A \to k_\alpha$ into a real closed field k_α. We may then define $\mathscr{O}(\alpha)$ to be the point in RCVR-spec A represented by the composition $A \to k_\alpha \to \mathscr{O}(k_\alpha)$. This defines a map $\mathscr{O}$: RCF-spec $A \to$ RCVR-spec A which is independent of the choice of k_α and which is clearly functorial. Since the condition "$a \mid b$" reduces to "$a \neq 0 \vee b = 0$" in the case of a field, this map is continuous in the constructible topologies. But this also shows *that $\mathscr{O}$ is not continuous in any of the spectral topologies induced by using strict or weak inequalities with 'divides' or 'not divides'.* A second view is given by examining specialization in RCVR-spec $\mathscr{O}(\mathscr{F})$. Let us look at the points in RCVR-spec $\mathscr{O}(\mathscr{F})$ in detail:

First, we can map $\mathscr{O}(\mathscr{F})$ identically to itself. This represents a point $\alpha \in$ RCVR-spec $\mathscr{O}$. Secondly, we can embed $\mathscr{O}(\mathscr{F})$ in R and then embed R in a real valuation subring $\mathscr{O}(R)$ as done for $\mathscr{F}$. This represents a point $\beta \in$ RCVR-spec $\mathscr{O}(\mathscr{F})$. Finally, we can define a third point γ by projecting $\mathscr{O}(\mathscr{F})$ onto $\mathscr{F}$ and embedding $\mathscr{F}$ in a real closed valuation ring, for example $\mathscr{O}(\mathscr{F})$ itself.

Looking at specializations in the spectral topology defined by using '$\leq$' and '$\mid$' as positive predicates, we have $\alpha \to \beta$: One easily checks that $\geq$ and $\mid$ are preserved. We also have $\alpha \to \gamma$. To check this, note that if $x \mid y$ in $\mathscr{O}(\mathscr{F})$, then either both x and y are in $\mathfrak{m}$, in which case $\gamma(x) = \gamma(y) = 0$ and $\gamma(x) \mid \gamma(y)$, or neither is in $\mathfrak{m}$, and again $\gamma(x) \mid \gamma(y)$. The canonical place $R \to \mathscr{F}$ *should* be represented by a specialization $\beta \to \gamma$. Furthermore, β and γ really represent points in the real spectrum of $\mathscr{O}(\mathscr{F})$, where γ specializes β. But if $x \in \mathfrak{m} \setminus \{0\}$ and $y \in \mathscr{O}(\mathscr{F}) \setminus \mathfrak{m}$, then $\beta(x) \mid \beta(y)$ in R but $\gamma(x) \nmid \gamma(y)$ in $\mathscr{O}(\mathscr{F})$.

The above is one reason for objecting to the standard 'divides' predicate. A second, more general objection involves the question of whether certain sheaves of functions turn out to have stalks which are local rings, see Proposition 7.5 ff.

We can fix all these problems by the artifice of introducing a slightly different 'divides' predicate, '$\|$', governed by the axiom

$$\text{DNU} : \forall X, Y (X \| Y \Leftrightarrow \exists Z (XZ = Y) \wedge \forall Z (\neg YZ \neq 1)).$$

This can be read as "X divides the non-unit Y". We let $\mathscr{L}$ be the language $\mathscr{L}\langle 0, 1, +, \times, \leq, \| \rangle$, which is still a conservative extension of the language of ordered rings. *From now on we consider DNU, not DIV, to be part of the theory RCVR.*

Note that DNU implies that x is not a unit if and only if $1 \| x$. This has two useful consequences. First, suppose R and L are two RCVR's, $S \subset R$ is a subring, $f : S \to L$ is an $\mathscr{L}$-homomorphism, and $s \in S$ is a non-unit *in R*. Then $f(s)$ is a non-unit in L. Second, if R is an RCVR, the old 'divide' predicate may be defined by $a \mid b \Leftrightarrow a \| b \vee (\neg 1 \| b \wedge \neg 1 \| a)$. Thus RCVR admits quantifier elimination in $\mathscr{L}$.

We pause to point out the effect this has on specialization in the example RCVR-spec $\mathcal{O}(\mathscr{F})$ above. Suppose $x, y \in \mathcal{O}(\mathscr{F})$. If $\beta(x) \parallel \beta(y)$, then $y=0$. In this case $\alpha(x) \parallel \alpha(y)$. If $\alpha(x) \parallel \alpha(y)$, then $y \in \mathfrak{m}$ and $\gamma(x) \parallel \gamma(y)$. From this we see that $\beta \to \alpha \to \gamma$. Thus the place $R \to \mathscr{F}$ gives a specialization which 'factors through' the associated valuation ring.

We now show that RCVR has enough prime models. This is tantamount to understanding what points in RCVR-spec A look like for a commutative ring A with 1.

Let $\alpha: A \to R$ be a CRO-homomorphism into an RCVR and let $\mathfrak{p}_\alpha$ be the kernel of α. Then R induces a total ordering on $A/\mathfrak{p}_\alpha$. Let F_α be the real closure of the quotient field QF$(A/\mathfrak{p}_\alpha)$ with respect to this ordering. Then $F_\alpha \subset \mathrm{QF}(R)$, and $R \cap F_\alpha$ is a convex real closed subring of F_α. Set $\mathcal{O}_\alpha = R \cap F_\alpha$. Clearly any RCVR L containing an image of A and defining the same point in RCVR-spec A contains an $\mathscr{L}$-isomorphic copy of $\mathcal{O}_\alpha$. Since division in $\mathcal{O}_\alpha$ agrees with that in R, either $\mathcal{O}_\alpha$ is a field or an elementarily embedded RCVR. In the latter case $k_\alpha = \mathcal{O}_\alpha$ and in the former case $k_\alpha = \mathcal{O}(\mathcal{O}_\alpha)$.

Note that RCVR is a rigid theory since an algebraic automorphism of an RCVR induces one on its quotient field which is real closed.

Having obtained representations of points in RCVR-spec A, we need to understand specialization. To do this we consider two distinct points α and β such that β specializes α (written $\alpha \to \beta$). By considering the $\geq$ relation, we obtain an order-preserving homomorphism $A/\mathfrak{p}_\alpha \to A/\mathfrak{p}_\beta$ – i.e., a specialization in the real spectrum. By replacing A by $\alpha(A)$ it suffices to consider the situation where $\mathfrak{p}_\alpha = (0)$ and α is the identity on A. Letting $\mathfrak{p} = \mathfrak{p}_\beta$, $\mathcal{O} = \mathcal{O}_\alpha$, and $F = F_\alpha = \overline{\mathrm{QF}(A)}$, α and β may be represented by the following diagram, where $\mathfrak{p}$ is a convex prime in A:

$$\begin{tikzcd}
F \arrow[d, dash] & F_\beta \arrow[d, dash] \\
\mathcal{O} \arrow[d, dash] & \mathcal{O}_\beta \arrow[d, dash] \\
A \arrow[r] & A/\mathfrak{p}_\beta .
\end{tikzcd}$$

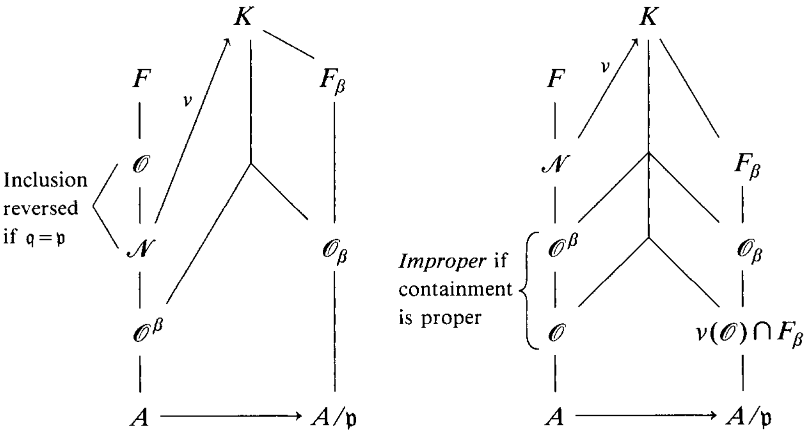
If $\mathcal{O} = \mathcal{O}(F)$, we ignore this and identify $\mathcal{O}$ with F.

Now let $\mathfrak{q}$ be the center of $\mathcal{O}$ on A. *Suppose that* $\mathfrak{q} \subset \mathfrak{p}$. Then if $a \in A$ is a non-unit in $\mathcal{O}$, $a \in \mathfrak{q}$ and hence $\beta(a) = 0$. It follows that $\parallel$ is automatically respected by β and the above diagram leads to a specialization. These are not the only specializations as we will see in a minute, but we first show a more convenient way to represent these.

Given $\mathfrak{p}$ there is a largest RCVR in F with center $\mathfrak{p}$ on A. Call this $\mathcal{N}$, let $\mathfrak{n}$ be its maximal ideal, and let K be its residue class field. Then there is a real place $v:\mathcal{N}\to K$ and F_β is a real closed subfield of K. We can pull $\mathcal{O}_\beta$ back to $v^{-1}(\mathcal{O}_\beta)\subset \mathcal{N}\subset\mathcal{O}\subset F$, and since all maps are order-preserving and $\mathcal{O}_\beta$ is convex, $v^{-1}(\mathcal{O}_\beta)$ is a convex subring of F and hence itself an RCVR. We denote $v^{-1}(\mathcal{O}_\beta)$ by $\mathcal{O}^\beta$ and its maximal ideal by $\mathfrak{m}^\beta$.

Fixing $\mathfrak{p}$ it is straightforward to verify that distinct (proper) RCVR's in F_β yield distinct RCVR's $\mathcal{O}^\beta$ in F with $\mathfrak{p}\subsetneq\mathfrak{m}^\beta\cap A$. Furthermore, an RCVR in F_β is completely determined by its intersection with QF$(A/\mathfrak{p})$ because F_β is algebraic over this quotient field. From this it is easily verified that every pair $(\mathcal{O}^\beta,\mathfrak{p})$ with $\mathcal{O}^\beta\subset\mathcal{O}$ and $\mathfrak{q}\subset\mathfrak{p}\subsetneq\mathfrak{m}^\beta\cap A$ yields a distinct specialization of α.

Pulling back F_β itself gives $\mathcal{O}^\beta=\mathcal{N}$. This really corresponds to the map $\beta:A\to\mathcal{O}(F_\beta)$. Perhaps unfortunately, any other RCVR in F with center $\mathfrak{p}$ yields the same point β. With this inaccuracy, we have generated a family $\{(\mathcal{O}^\beta,\mathfrak{p})\mid\mathcal{O}^\beta\subset\mathcal{O},\ \mathfrak{q}\subset\mathfrak{p}\subsetneq\mathfrak{m}^\beta\cap A\}$ of specializations of α. We will call these *proper specializations*. The situation discussed is given by the left side of diagram 1.



Case $\mathfrak{q}\subset\mathfrak{p}$ Case $\mathfrak{p}\subsetneq\mathfrak{q}$

Diagram 1

We now have to treat the possibility that $\mathfrak{p}\subsetneq\mathfrak{q}$. In this case we construct $\mathcal{O}^\beta$ as before and claim that $\mathfrak{m}^\beta\cap A=\mathfrak{q}$. For suppose that $\mathfrak{m}^\beta\cap A\subsetneq\mathfrak{q}$. Then there is an $a\in A$ which is a non-unit in $\mathcal{O}$ but such that $\beta(a)$ is a unit in $\mathcal{O}_\beta$, so $\alpha\not\to\beta$. On the other hand, if $\mathfrak{q}\subsetneq\mathfrak{m}^\beta\cap A$, then we choose $a\in\mathfrak{m}^\beta\cap A$, $a\notin\mathfrak{q}$, and $b\in\mathfrak{q}$, $b\notin\mathfrak{p}$. Then $ab\parallel b$ in $\mathcal{O}$ but $\beta(ab)\not\parallel\beta(b)$, so again $\alpha\not\to\beta$.

Thus $\mathcal{O}^\beta$ and $\mathcal{O}$ have the same centers on A. Since $\mathfrak{p}\subsetneq\mathfrak{q}$, $\mathcal{O}\subset\mathcal{N}$ and we may apply v to $\mathcal{O}$. It is then easy to see that we must have $v(\mathcal{O})\cap F_\beta\subset\mathcal{O}_\beta$ in order to preserve $\parallel$, and if this containment holds we always get a specialization. We thus have a second family of specializations consisting of pairs $(\mathcal{O}^\beta, \mathfrak{p})$ with $\mathcal{O} \subset \mathcal{O}^\beta$ and $\mathfrak{p} \subsetneq \mathfrak{m}^\beta \cap A = \mathfrak{q}$. We consider specializations of this type with $\mathcal{O}^\beta = \mathcal{O}$ to be proper but those with $\mathcal{O} \subsetneq \mathcal{O}^\beta$ to be *improper*.

The reason for the word 'improper' is that it is precisely the improper specializations which prevent RCVR from having open projections in the language we have chosen. In general, we have

Lemma 4.2. *Let* $A \in \mathrm{CRO}$ *and let* $\pi : \text{RCVR-spec}\, A[T] \to \text{RCVR-spec}\, A$ *be as defined in Lemma* 3.1. *Then* π *is an open map if and only if there are no improper specializations in* RCVR-spec A.

Proof. π is open if and only if given a *closed* formula $\varphi(T, \bar{a}) \in \mathcal{L}_A$, then $\forall T\varphi(T, \bar{a})$ is also closed. Since $\mathcal{X}(\forall T\varphi(T, \bar{a}))$ is constructible, it is closed if and only if it is closed under specialization.

First let us consider a point $\alpha \in \mathcal{X}(\forall T\varphi(T, \bar{a}))$ and a proper specialization $\alpha \to \beta$ as represented by the left side of Diagram 1 or by the right side with $\mathcal{O}^\beta = \mathcal{O}$. Thus $\mathcal{O} \models \forall T\varphi(T, \bar{a})$ and $\mathcal{O}^\beta \subset \mathcal{O}$. Hence $\mathcal{O}^\beta \models \forall T\varphi(T, \bar{a})$. It follows that $\nu(\mathcal{O}^\beta) \models \forall T\varphi(T, \nu(\bar{a}))$ since φ is closed, and, since $\nu(\bar{a}) = \beta(\bar{a})$, we have $\mathcal{O}^\beta \models \forall T\varphi(T, \beta(\bar{a}))$. Thus $\beta \in \mathcal{X}(\forall T\varphi(T, \bar{a}))$.

This shows the 'if' part. To get the converse we assume there is an improper specialization and consider the situation on the right in Diagram 1 with $\mathcal{O} \subsetneq \mathcal{O}^\beta$. Note that $\mathfrak{q} \neq 0$ because $\mathfrak{p} \subsetneq \mathfrak{q}$. Thus there must be $a, b \in A$ such that $a/b \in \mathcal{O}^\beta - \mathcal{O}$, and by multiplying by a non-zero element of $\mathfrak{q}$ we may find such $a, b \in \mathfrak{q}$. But then $\mathcal{O} \models \forall T(a - bT \geq 0)$ and $\mathcal{O}^\beta \not\models \forall T(a - bT \geq 0)$. Since $\alpha \to \beta$, it follows that $\mathcal{X}(\forall T(a - bT \geq 0))$ is not a closed. Thus $\forall T(U - VT \geq 0)$ is not RCVR-closed even though "$U - VT \geq 0$" is an RCVR-closed formula. □

Recall that a ring A is a *real Prüfer ring* if every localization of A at a prime is a real valuation ring. (For example, a real valuation ring itself is Prüfer.) If A is real Prüfer, then so is every homomorphic image of A. Furthermore, every real valuation overring of A in its quotient field is simply a localization and is therefore completely determined by its center. Putting this together, we see there are no improper specializations in RCVR-spec A.

Corollary 4.3. *Let* $\pi : \text{RVCR-spec}\, A[T] \to \text{RCVR-spec}\, A$ *be as defined in Lemma* 3.1. *Then* π *is an open mapping if* A *is real Prüfer ring.* □

We can give a nice picture of specialization in the case where A is a real Prüfer ring:

Proposition 4.4. *Let* A *be real Prüfer and let* $\alpha \in \text{RCVR-spec}\, A$. *Then the specializations of* α *correspond to pairs* $(\mathfrak{p}_\beta, \mathfrak{m}_\beta)$ *of convex primes in* $\alpha(A)$ *with* $\mathfrak{q} \subset \mathfrak{m}_\beta$ *and* $\mathfrak{p}_\beta \subset \mathfrak{m}_\beta$. *Corresponding prime models may be obtained by forming*

$(\alpha(A)/\mathfrak{p}_\beta)_{(\mathfrak{m}_\beta/\mathfrak{p}_\beta)}$ *and applying* $\mathcal{O}$ *if* $\mathfrak{m}_\beta = \mathfrak{p}_\beta$. *If* β_1 *and* β_2 *are two specializations of* α, *then* $\beta_1 \to \beta_2$ *if and only if* $\mathfrak{p}_{\beta_1} \subset \mathfrak{p}_{\beta_2}$ *and* $\mathfrak{m}_{\beta_1} \subset \mathfrak{m}_{\beta_2}$. $\square$

Corollary 4.5. *Let A be Prüfer, let* $\alpha \in \text{RCVR-spec}\, A$, *and let L be the set of convex primes of* $\alpha(A)$ *containing* $\mathfrak{q}$, *linearly ordered by inclusion. Partially order* $L \times L$ *with* $(x, y) \le (x', y')$ *if* $x \le x'$ *and* $y \le y'$. *Then* $\{\beta \in \text{RVCR-spec}\, A \mid \alpha \to \beta\}$, *partially ordered by specialization, is isomorphic to* $\{(x, y) \in L \times L \mid x \le y\}$. *It is totally ordered if and only if* $\#(L) \le 2$. $\square$

For an application, see Lemma 5.5.

We end this section by pointing some connections between RCVR-spec and RCF-spec.

Proposition 4.6. *Let A be any commutative ring with* 1. *Define* $\mathcal{O} : \text{RCF-spec}\, A \to \text{RCVR-spec}\, A$ *by embedding* k_α *in* $\mathcal{O}(k_\alpha)$. *Define* $q : \mathcal{O} : \text{RCVR-spec}\, A \to \text{RCF-spec}\, A$ *by sending* k_α *to its quotient field* F_α. *Then* $\mathcal{O}$ *and q are functorial,* $\mathcal{O}$ *is a homeomorphism onto its image if the same types of topologies are used for* RCF-spec *and* RCVR-spec, *and* $q \circ \mathcal{O} = \text{id}$. *The image of* $\mathcal{O}$ *is dense in the spectral topology and is a spectral retract of* RCVR-spec A.

Proof. As previously mentioned, $\mathcal{L}$ contains and can be translated into the language of ordered rings. Furthermore, the open predicate $\neg x \| y$ is RCF-equivalent to $y \neq 0$. It follows easily that $\mathcal{O}$ is a homeomorphism onto its image. Also, $q^{-1}(\mathscr{X}(a \ge 0)) = \mathscr{X}(a \ge 0)$ and $q^{-1}(\mathscr{X}(a > 0)) = \mathscr{X}(a > 0)$ – in other words, $q^{-1}(\mathscr{X}_{\text{RCF}}(\varphi)) = \mathscr{X}_{\text{RCVR}}(\varphi)$. Thus q is continuous. Since $F_{\mathcal{O}(k_\alpha)}$ contains k_α, $q \circ \mathcal{O} = \text{id}$.

If $\beta \in \text{RCVR-spec}\, A$, the point defined by $\mathcal{O}(\text{QF}(k_\beta))$ is always a generalization of β. This shows density. Functoriality is immediate from the definitions.

Finally, $\beta \mapsto \mathcal{O}(\mathscr{F}_\beta)$ defines a (functorial) map $\text{RCVR-spec}\, A \to \text{RCVR-spec}\, A$. It is continuous in the spectral topology. $\square$

5. Affine space

For this section $\mathscr{T}$ will be a fixed theory with enough prime models, R a fixed model of $\mathscr{T}$, and $\{k_\alpha\}_{\alpha \in \mathscr{T}\text{-spec}\, R}$ a set of prime models. This set may or may not contain just one k_α. We will always identify points of $\mathscr{T}$-spectra with canonical maps into prime models. To simplify notation later, we let

$$A_n = A_n(R) = R[T_1, \ldots, T_n]$$

be the polynomial ring in n indeterminates over R.

We define the *affine line* $\mathbb{A}^1$ to be the disjoint union $\dot{\coprod} k_\alpha$ of the k_α's, and we let $\pi : \mathbb{A}^1 \to \mathscr{T}\text{-spec}\, R$ be the projection sending k_α to α. Thus $\mathbb{A}^1$ is fibred over $\mathscr{T}$-spec R.

Definition 5.1 By *n-dimensional affine space over R* we mean the n-fold fibred product of $\mathbb{A}^1$. It is denoted by $\mathbb{A}^n(R)$, or just $\mathbb{A}^n$, and is isomorphic to $\dot{\coprod} k_\alpha^n$. We denote the fibring by π, so $\pi(x_1,\dots,x_n)=\alpha$ if $(x_1,\dots,x_n)\in k_\alpha^n$.

The inclusion $\iota: R \hookrightarrow A_n$ induces a fibring $\Pi: \mathscr{T}\text{-spec}\, A_n \to \mathscr{T}\text{-spec}\, R$ with $\Pi=\iota^*$. If $\bar{x}=(x_1,\dots,x_n)\in\mathbb{A}^n$ and $\pi(\bar{x})=\alpha$, then $\bar{x}$ defines a unique CRO-homomorphism $\bar{x}: A_n \to k_\alpha$ with $\bar{x}(a)=\alpha(a)$ for $a\in R$ and $\bar{x}(T_i)=x_i$. This defines a fibred inclusion $j:\mathbb{A}^n \hookrightarrow \mathscr{T}\text{-spec}\, A_n$. We may thus consider $\mathbb{A}^n$ to be a (fibred) subset of $\mathscr{T}\text{-spec}\, A_n$.

Let $X\subset\mathbb{A}^n$ and suppose there is a formula $\theta(T_1,\dots,T_n)$ in $\mathscr{L}$ such that $X=\{\bar{x}\in\mathbb{A}^n \mid k_{\pi(\bar{x})}\models\theta(x_1,\dots,x_n)\}$. Then X is called *constructible*. The set of constructible subsets of $\mathbb{A}^n$ is denoted by $\mathscr{C}(\mathbb{A}^n)$.

Equivalently, $X\subset\mathbb{A}^n$ is constructible if it is the restriction to $\mathbb{A}^n$ of a subset of $\mathscr{T}\text{-spec}\, A_n$ which is open in the constructible topology. Thus $\mathscr{C}(\mathbb{A}^n)$ is the weakest restricted topology on $\mathbb{A}^n$ making j continuous. It is called the constructible topology. Note that $\pi:\mathbb{A}^n\to\mathscr{T}\text{-spec}\, R$ is continuous in the constructible topologies.

Similarly, we define $\mathscr{C}^\circ(\mathbb{A}^n)$ to be the set of restrictions of sets in $\mathscr{C}^\circ(\mathscr{T}\text{-spec}\, A_n)$ to $\mathbb{A}^n$, or those $X\in\mathscr{C}(\mathbb{A}^n)$ defined by open formulae. This is also a (restricted) topology known as the *weak topology*.

If a constructible subset of $\mathscr{T}\text{-spec}\, A_n$ is open in the spectral topology, then by compactness it is in $\mathscr{C}^\circ(\mathscr{T}\text{-spec}\, A_n)$. But there are many more open sets in the spectral topology, which is a topology in the classical sense. The restriction of these sets to $\mathbb{A}^n$ yields a topology on $\mathbb{A}^n$ called the *strong topology*.

Since Π is continuous in the spectral topologies, the fibring $\pi:\mathbb{A}^n\to\mathscr{T}\text{-spec}\, R$ is continuous if $\mathbb{A}^n$ has the strong topology and $\mathscr{T}\text{-spec}\, R$ has the spectral topology. In particular, $\mathbb{A}^n$ is a fibred product in TOP. For the weak topology we have:

Lemma 5.2. *If $\mathbb{A}^n$ is endowed with the weak topology and $\mathscr{T}$-spec R with the spectral topology, then π is continuous if and only if every open subset of $\mathscr{T}$-spec R is constructible. In particular, this happens if $\mathscr{T}$-spec R is finite.*

Proof. Suppose π is continuous and $U\subset\mathscr{T}\text{-spec}\, R$ is open. Then $\pi^{-1}(U)$ is defined by a formula $\theta(\bar{T})$. Thus $\alpha\in U \Leftrightarrow k_\alpha\models\exists\bar{T}\theta(\bar{T})$, so U is constructible. The rest is clear. $\square$

We can prove two more general results about affine space. The first is an 'Artin-Lang' property. Restriction induces a surjection $\varrho:\mathscr{C}(\mathscr{T}\text{-spec}\, A_n)\to\mathscr{C}(\mathbb{A}^n)$ sending $\mathscr{C}^\circ(\mathscr{T}\text{-spec}\, A_n)$ to $\mathscr{C}^\circ(\mathbb{A}^n)$.

Proposition 5.3. *ϱ is an isomorphism of boolean algebras. Thus $\mathscr{T}$-spec A_n is the Stone space of $\mathscr{C}(\mathbb{A}^n)$ and points correspond to prime filters in $\mathscr{C}^\circ(\mathbb{A}^n)$.*

Proof. It suffices to show that if $X\neq\emptyset$ is constructible in $\mathscr{T}\text{-spec}\, A_n$, then $\varrho(X)\neq\emptyset$. Let $\beta\in X$, $X=\mathscr{X}(\theta(\bar{T}))$, and $\alpha=P(\beta)$. Then $k_\beta\models\exists\bar{T}\theta(\bar{T})$ and $k_\alpha\subset k_\beta$ as an

$\mathscr{L}_R$-substructure. It follows that $k_\alpha \models \exists \bar{T}\theta(\bar{T})$, and this yields a point $\bar{x} \in \varrho(X)$ with $\pi(\bar{x}) = \alpha$. □

The second general result is that projection is an open mapping if our theory has open projections. If $m \geq n$, then the inclusion $A_n \hookrightarrow A_m$ yields a projection $\mathrm{pr}: \mathscr{T}\text{-spec}\, A_m \to \mathscr{T}\text{-spec}\, A_n$. If $U \in \mathscr{C}^\circ(\mathscr{T}\text{-spec}\, A_m)$ is open, then $\mathrm{pr}(U)$ is given by a formula of the form $\exists T_{n+1}, \dots, T_m \theta(T_1, \dots, T_n, T_{n+1}, \dots, T_m)$ with θ open. By definition, $\mathrm{pr}(U)$ is open. Of course, pr is fibred and induces a projection $\mathrm{pr}: \mathbb{A}^m \to \mathbb{A}^n$ which is just projection onto the first n coordinates in each fiber. We have

Proposition 5.4. *Let* pr *be projection from* $\mathscr{T}$-spec A_m *to* $\mathscr{T}$-spec A_n *or from* $\mathbb{A}^m$ *to* $\mathbb{A}^n$. *If* $\mathscr{T}$ *has open projections, then* pr *is an open continuous fibred map as long as both spaces are endowed with the same type of topology.* □

We now turn to the examples.

ACF-spec is the Zariski spectrum. Thus R is an algebraically closed field and ACF-spec R is just a single point. $\mathbb{A}^n$ is R^n. The weak and strong topologies coincide and are the usual Zariski topology. The constructible topology is the topology of all constructible subsets. ACF-spec A_n with the spectral topology is the usual 'affine n-space' in the more abstract sense.

RFC-spec is the real spectrum. So if R is a real closed field, RCF-spec R is a point, and $\mathbb{A}^n = R^n$. The constructible subsets are the semialgebraic sets and weakly open sets are open semialgebraic sets. The strong topology is the usual topology on R^n, since open balls are open constructible. If we use RCFS-spec instead, all semialgebraic subsets are weakly open and the strong topology on R^n is totally disconnected.

RCF-spec A_n is well-understood, see [1]. For later comparison we name the points of RCF-spec A_1, writing $R[T]$ for A_1. First, there are rational points $\{a \mid a \in R\}$. These constitute the image $j(R)$. For them $k_a \cong R$. For the rest, the prime model is the real closure of $R(T)$ with respect to a total order. T defines a Dedekind cut of R which either is some $a \in R$, in which case we get a point a^+ if $T > a$ and a point a^- if $T < a$, or is not in R, in which case we get a point β. If $|T|$ is bounded by an element of R, the point is finite. If not, then we get the point $\beta = \infty$ or $\beta = -\infty$, according to the sign of T. The only proper specializations are $a^+ \to a$ and $a^- \to a$.

RCVR-spec R *never* consists of a single point for a real closed valuation ring R, although it does for a real closed field. Let α be the point represented by the inclusion into $\mathscr{O}(\mathrm{QF}(R))$. This is the generic point. Since $\mathfrak{m}_\alpha = \mathfrak{p}_\alpha = 0$, and the ordering on R is determined by squares in $\mathrm{QF}(R)$, all points in RCVR-spec R are specializations of α. If R has rank n (i.e. if the value group of the associated real valuation has rank n), then the convex prime ideals of R form a chain of length $n+1$. Proposition 4.6 shows that the only point with no specializations, i.e. the only closed point, is the point ω with $\mathfrak{m}_\omega = \mathfrak{p}_\omega = \mathfrak{m}$ (the maximal ideal of R). Using Corollary 4.5, we get

Lemma 5.5. *Let R be an* RCVR. *The partial order induced by specialization has an initial point α and a final point ω. If $n = \mathrm{rk}(R)$, there are $(n+1)(n+2)/2$ points. They are linearly ordered if and only if $n = 1$.* $\square$

We now describe the points of RCVR-spec A_n in the simplest case, namely where $R = \mathcal{O}(\mathcal{F})$ for a real closed field $\mathcal{F}$. We have already seen that RCVR-spec R consists of three points which we will (re-)label α, μ, and ω. μ is represented by the identity on R. Consider the fibre $\Pi^{-1}(\alpha)$. A point β in this fibre is a map $\beta : A_n \to L$ into and RCVR L such that $\operatorname{Ker} \beta \cap R = 0$ and $\beta(R - \{0\}) \subset L^\times$. Thus β has a unique extension to $A_n(\mathrm{QF}(R)) = \mathrm{QF}(R)[T_1, \ldots, T_n]$. Hence $\Pi^{-1}(\alpha)$ is RCVR-spec$(A_n(\mathrm{QF}(R)))$ as a space with any subspace topology. Similarly, $\Pi^{-1}(\omega) \cong$ RCVR-spec$(A_n(\mathrm{QF}(R/\mathfrak{m})))$. This motivates a study of RCVR-spec of the affine coordinate ring over a real closed *field*. Finally, a point $\beta \in \Pi^{-1}(\mu)$ is represented by $\beta : A_n \to L$ such that $\operatorname{Ker} \beta \cap R = 0$ but $\beta(R)$ is elementarily embedded in L. Using the ordering β defines a finite point in the real spectrum of $A_n(R)$ or, equivalently, of $A_n(\mathrm{QF}(R))$, and hence an imbedding of $\mathrm{QF}(R)$ into a real closure k of $\mathrm{QF}(A_n(R))$. To determine β completely we need to pick an RCVR in k whose intersection with $\mathrm{QF}(R)$ is R. There are, in general, many possibilities. We do not go into detail on specializations.

As for $\mathbb{A}^n$, we at least point out that $\mathrm{QF}(R/\mathfrak{p})$ is contained in $\mathcal{O}(\mathrm{QF}(R/\mathfrak{p}))$ which lies in RCVR-spec R for every convex prime $\mathfrak{p} \subset R$. Thus $\mathrm{QF}(R/\mathfrak{p})^n$ lies in a fibre of $\mathbb{A}^n$. In this fibre divisibility plays no role, so the induced subspace topologies agree with the standard ones.

We now turn to the problem of describing RCVR-spec$(R[T_1, \ldots, T_n])$ for a real closed field R. RCVR-spec R consists of a single point with prime model $\mathcal{O}(R)$, and RCVR-spec$(R[T_1, \ldots, T_n])$ is fibred over this point. Proposition 4.6 gives us a fibred homeomorphism of RCF-spec$(R[T_1, \ldots, T_n])$ onto a subspace which is dense in the spectral topology. We may thus view R^n as a (fibred) subspace of RCVR-spec$(R[T_1, \ldots, T_n])$. The induced subspace topologies are the constructible and strong topologies (viewing R^n as $\mathbb{A}^n$ in RCF).

We explicitly describe the points in RCVR-spec $R[T]$ for a real closed field R. We identify points of $\alpha \in$ RCF-spec $R[T]$ with their images under $\mathcal{O}$. If α is not rational, then $\mathfrak{p}_\alpha = 0$. If $\alpha = a^\pm$, then $\mathfrak{m}_\alpha = \langle T - a \rangle$ and Proposition 4.4 yields a specialization $a_v^\pm$ of $a^\pm$ corresponding to the rank-one real valuation ring defined by the convex closure of $R[T]$ in $R(T)$. If T gives a non-rational Dedekind cut, then $\mathfrak{m}_\alpha = 0$ and there are no new points. All proper specializations are the chains $a^+ \to a_v^+ \to a$ and $a \to a_v \to a$.

6. Definable sections

In this section we introduce the notion of a definable function on the $\mathcal{T}$-spectrum of an arbitrary commutative ring A with 1. Our treatment is based on work of

Brumfiel and Schwartz [11] for the real spectrum. We end up with sheaves of functions which serve as structure sheaves in the context of semialgebraic topology.

Let $\mathrm{pr}: \mathscr{T}\text{-spec}\, A[T] \to \mathscr{T}\text{-spec}\, A$ be the projection induced by the inclusion of A into the polynomial ring in one indeterminate over A. By a *section* over a set $X \subset \mathscr{T}\text{-spec}\, A$ we mean a function $s: X \to \mathscr{T}\text{-spec}\, A[T]$ such that $\mathrm{pr}(s(\alpha)) = \alpha$ for all $\alpha \in X$. If s is a section we let $\Gamma(s) = s(X)$ (the 'graph' of s).

Suppose a section s over a constructible set X has a constructible graph $\Gamma(s)$, let $\theta(T) \in \mathscr{L}_{A[T]}$ define $\Gamma(s)$, and let $\psi \in \mathscr{L}_A$ define X. We say that s is *definable* if

$$\mathscr{T}_{A[T]} \cup \psi \vdash \exists! T\theta(T).$$

A section over an arbitrary X is definable if it is locally definable in the constructible topology.

Given $\alpha \in \mathscr{T}\text{-spec}\, A$, let $\mathscr{T}_\alpha$ be the complete theory of k_α. The fact that s is a section implies that $\mathscr{T}_\alpha \cup \mathscr{T}_{A[T]} \cup \psi \vdash \exists! T\theta(T)$, for otherwise $\Gamma(s)$ would have at least two points in some fibre $\mathrm{pr}^{-1}(\alpha)$. For rigid theories, such as RCF and RCVR, it follows that any section with constructible graph is definable, but this is not the case for a theory such as ACF. As an example, let $A = \mathbb{R}$. Then ACF-spec R is a single point, and the constructible set in ACF-spec $\mathbb{R}[T]$ defined by $T^2 = -1$ also consists of a single point. The only model of $\mathrm{ACF} \cup \mathscr{L}_{\mathbb{R}[T]} \cup T^2 = -1$ is $\mathbb{C}$, but $\mathbb{C}$ has two points whose square is -1.

Let $\mathscr{D}(X)$ denote the set of all definable sections over X. Since $+$ and $\times$ are always in $\mathscr{L}$, and sums and products are unique, it is easy to see that $\mathscr{D}(X)$ is always a ring and that $\mathscr{D}$ defines a sheaf of commutative rings with 1 on $\mathscr{T}\text{-spec}\, A$ in the constructible topology.

Lemma 6.1. *Suppose $s \in \mathscr{D}(X)$, $\alpha \in X$, and $s(\alpha) = \beta$. Then β is represented by a homomorphism $\beta: A[T] \to k_\alpha$, so $k_{s(\alpha)} \cong k_\alpha$.*

Proof. We may assume X is constructible. Since $\mathrm{pr}(\beta) = \alpha$, we have a commutative diagram

$$\begin{array}{ccc} A[T] & \xrightarrow{\beta} & k_\beta \\ \uparrow & & \uparrow \\ A & \xrightarrow[\alpha]{} & k_\alpha \end{array}$$

with k_α $\mathscr{L}_A$-elementary embedded in k_β. Suppose $\Gamma(s)$ is defined by the formula $\theta(T) \in \mathscr{L}_{A[T]}$. Let $\Theta(T)$ be the formula "$\exists! T\theta(T)$" ($\exists!$ means "there exists a unique"). Then Θ is a sentence in $\mathscr{L}_A$ and $k_\beta \vDash \Theta$, with witness $\beta(T)$. Thus $k_\alpha \vDash \Theta$, so $\beta(T) \in k_\alpha$. $\square$

We interpret definable sections as *functions* $s: \mathscr{T}\text{-spec}\, A \to \dot{\coprod} k_\alpha$ where $\{k_\alpha\}_{\alpha \in \mathscr{T}\text{-spec}\, A}$ is a set of prime models and $s(\alpha) \in k_\alpha$.

Proposition 6.2. *Let $\mathscr{A}$ be the ring $\mathscr{D}(\mathscr{T}\text{-spec}\, A)$ of global definable sections, and let $(\operatorname{Spec} \mathscr{A}, \mathfrak{Z})$ be the Zariski spectrum of $\mathscr{A}$ with the canonical structure sheaf. Then there is a map $z: \operatorname{Spec} \mathscr{A} \to \mathscr{T}\text{-spec}\, A$ such that $z_* \mathfrak{Z}(X) = \mathscr{D}(X)$ for every $X \subset \operatorname{Spec} \mathscr{A}$ and which is a homeomorphism if $\mathscr{T}\text{-spec}\, A$ is given the constructible topology.*

Proof. If $s \in \mathscr{A}$, we view s as a function and define $Z(f) = \{\alpha \in \mathscr{T}\text{-spec}\, A \mid f(\alpha) \notin k_\alpha^\times\}$. Note that $Z(f)$ is constructible and that$Z(fg) = Z(f) \cup Z(g)$ as usual. Also note that the functions

$$\chi_f = \begin{cases} 1/f(\alpha) & \text{if } \alpha \notin Z(f), \\ 0 & \text{if } \alpha \in Z(f) \end{cases}$$

and

$$\chi_X = \begin{cases} 0 & \text{if } \alpha \notin X, \\ 1 & \text{if } \alpha \in X \end{cases}$$

for X constructible are definable.

Let $\mathfrak{p} \subset \mathscr{A}$ be a prime ideal. We define $z(\mathfrak{p}) = \{Z(f) \mid f \in \mathfrak{p}\}$. Using the above, and the observation that $Z(f \cdot \chi_f + g \cdot \chi_g) = Z(f) \cap Z(g)$, it is straightforward to check that $z(\mathfrak{p})$ is an ultrafilter in $\mathscr{C}(\mathscr{T}\text{-spec}\, A)$. Conversely, if $\alpha \in \mathscr{T}\text{-spec}\, A$ is an ultrafilter of constructible sets, then $\{f \mid Z(f) \in \alpha\}$ is prime. This defines an inverse for z.

If X is constructible, then $z^{-1}(X) = \{\mathfrak{p} \mid \chi_X \notin \mathfrak{p}\}$ is Zariski lopen. Thus z is continuous, and since both spaces are compact Hausdorff, z is a homeomorphism.

Assume X is constructible. Then $\mathfrak{Z}(z^{-1}(X)) \subset \mathscr{D}(X)$ as rings of functions. But if $s \in \mathscr{D}(X)$, we can extend s by zero to X^c and obtain a global definable section representing s. The proposition follows. □

Corollary 6.3. *$(\mathscr{T}\text{-spec}\, A, \mathscr{D})$ is a locally ringed space – in fact an affine scheme – and $\mathscr{D}$ is a flasque sheaf in the constructible topology.* □

An important example of definable sections are the *rational functions*. If $f \in A$, we associate f with the function $f(\alpha) = \alpha(f)$. We define a section $s \in \mathscr{D}(X)$ to be rational if X can be covered by constructible sets U_i such that, viewing s as a function, on each U_i we have $s(\alpha) = f_i(\alpha)/g_i(\alpha)$ for some $f_i, g_i \in A$ with $g_i(\alpha) \in k_\alpha^\times$ for all $\alpha \in U_i$. This yields a sheaf $\mathscr{R}$ of rings of rational functions, and $(\mathscr{T}\text{-spec}\, A, \mathscr{R})$ is a locally ringed space if all models of $\mathscr{T}$ are themselves local rings. In this case $\mathscr{D}$ is a sheaf of modules on $(\mathscr{T}\text{-spec}\, A, \mathscr{R})$.

Suppose we are given a CRO-homomorphism $f: A \to B$ and a section $s \in \mathscr{D}(\mathscr{T}\text{-spec}\, A)$. Viewing s as a function, we can compose with f^* to get a function $f_\# s : \mathscr{T}\text{-spec}\, B \to \dot{\coprod} k_{f^*\beta}$, where the disjoint union is taken over all $\beta \in \mathscr{T}\text{-spec}\, B$. Since $k_{f^*\beta} \subset k_\beta$, we can interpret this as a section over $\mathscr{T}\text{-spec}\, B$. Furthermore, if $\theta \in \mathscr{L}_{A[T]}$ defines s, then $f_\# s(\beta) = x$ if and only if $k_{f^*\beta} \vDash \theta(x)$, and this happens if and only if $k_\beta \vDash \theta(x)$, so $f_\# s \in \mathscr{D}(\mathscr{T}\text{-spec}\, B)$.

In this way we get a functorial CRO-homomorphism $f_{\#}:\mathscr{D}(\mathscr{T}\text{-spec}\,A)\to\mathscr{D}(\mathscr{T}\text{-spec}\,B)$. Applying the Zariski spectrum functor gives a functorial map of locally ringed spaces.

Proposition 6.4. *With respect to the constructible topology and the structure sheaf* $\mathscr{D}$, $\mathscr{T}$-spec *is a functor* CRO $\rightsquigarrow$ LRS *into the category of locally ringed spaces. If* $\{A_i\}$ *and* $\{f_{ij}:A_i\to A_j\}_{j\geq i}$ *is a filtered inductive system in* CRO, *then* $\mathscr{T}\text{-spec}(\varinjlim A_i)\cong\varprojlim\mathscr{T}\text{-spec}\,A_i$ *in* LRS.

Proof. In light of Proposition 6.2, it is enough to show that $\varinjlim\alpha_{i\#}:\varinjlim\mathscr{D}(\mathscr{T}\text{-spec}\,A_i)\to\mathscr{D}(\mathscr{T}\text{-spec}\varinjlim A_i)$ is an isomorphism, where $\alpha_i:A_i\to\varinjlim A_i$ denotes the canonical inclusion.

Let $s\in\mathscr{C}(\mathscr{T}\text{-spec}\varinjlim A_i)$. Then s is defined by a formula $\theta(T)\in\mathscr{L}\varinjlim A_i$. The fact that θ defines a section is equivalent to the fact that $k_\alpha\models\exists!\,T\theta(T)$ for all $\alpha\in\mathscr{T}\text{-spec}\varinjlim A_i$. By logical compactness we find a positive q.f. sentence $\sigma\in\Delta^+\varinjlim A_i$ such that $\mathscr{T}\vdash\sigma\Rightarrow\exists!\,T\theta(T)$. But we can find some i such that all constants in θ and σ are in $\alpha_j(A_j)$ for all $j\geq i$. Thus $\mathscr{T}_{A_j}\vdash\sigma$ and $\theta(T)$ defines a section $s_j\in\mathscr{D}(\mathscr{T}\text{-spec}\,A_j)$ for all $j\geq i$. These sections define an element of $\varinjlim\mathscr{D}(\mathscr{T}\text{-spec}\,A_i)$ mapping onto s, so we have shown surjectivity.

For injectivity, let $\beta_i:\mathscr{D}(\mathscr{T}\text{-spec}\,A_i)$ be the canonical inclusion and suppose we are given an element $s\in\mathscr{D}(\mathscr{T}\text{-spec}\,A_i)$ with $\varinjlim\alpha_{i\#}(s)=0$. Choose i such that $s=\beta_i(s_i)$ with $s_i\in\mathscr{D}(\mathscr{T}\text{-spec}\,A_i)$. Then $\alpha_{i\#}(s)$ is defined by a formula $\theta_i(T)$ defining the section s_i, and $\mathscr{T}\varinjlim A_i\models\theta_i(T)\Rightarrow T=0$. Again we find some $j\geq i$ such that $\mathscr{T}_{A_j}\models\theta_j(T)\Rightarrow T=0$. Thus $f_{ij\#}(s_j)=0$, and so $s=\beta_j(f_{ij\#}(s_i))=0$. □

Remark 6.5. The proof just given uses Proposition 6.2 to reduce to the case $X=\mathscr{T}\text{-spec}\,A$, but this is unnecessary. If $X\in\mathscr{C}(\mathscr{T}\text{-spec}\,A)$ and $s\in\mathscr{D}(X)$, then there is a formula $\psi\in\mathscr{L}_A$ defining X, a formula $\sigma\in\Delta^+A$, and a formula $\theta(T)\in\mathscr{L}_{A[T]}$ such that $\mathscr{T}\models\psi\wedge\sigma\Rightarrow\exists!\,T\theta(T)$ if we consider all the constants from A to be variables. Conversely, given any positive q.f. $\sigma(U_1,\ldots,U_n)\in\mathscr{L}$, any $\psi(\bar{U})\in\mathscr{L}$, and any $\theta(T,\bar{U})\in\mathscr{L}$ such that $\mathscr{T}\models\psi\wedge\sigma\Rightarrow\exists!\,T\theta(T)$, then if $\sigma(a_1,\ldots,a_n)\in\Delta^+A$ for $a_1,\ldots,a_n\in A$ we get a section $s\in\mathscr{D}(\mathscr{X}(\psi(a_1,\ldots,a_n)))$ defined by $\theta(T,a_1,\ldots,a_n)$.

Furthermore, if $f:A\to B$ is a CRO-morphism, substituting $f(a_i)$ for U_i defines the section $f_{\#}s$ over $f^{*-1}(X)$. This gives a model-theoretic description of $f_{\#}s$ which, when combined with the proof given above for the case $X=\mathscr{T}\text{-spec}\,A$, gives a direct proof of Proposition 6.4.

Next, we consider the rings A_n of Section 5. By Proposition 6.4, the fibrings are fibrings in LRS. We can also define a sheaf $\mathscr{D}$ on $\mathbb{A}^n$ whose sections over a constructible set $X\subset\mathbb{A}^n$ are simply fibred functions $f:X\to\mathbb{A}^1$ whose graphs are constructible subsets of $X\times\mathbb{A}^1\subset\mathbb{A}^n\times\mathbb{A}^1\cong\mathbb{A}^{n+1}$ determined by a defining formula. The main result is

Proposition 6.6. *The restriction* $j_{\#}$ *is an isomorphism.*

Proof. Let $X \in \mathscr{C}(\mathscr{T}\text{-spec}\, A_n)$. As in Proposition 5.3, define $\varrho(X)$ to be $X \cap \mathbb{A}^n$. Let $f \in \mathscr{D}(\varrho(X))$ be a fibred function with $\Gamma(f)$ defined by $\theta(T_1, \dots, T_{n+1})$. Then $\varrho(X)$ is contained in the set defined by $\exists!\, T_{n+1}\, \theta(T_1, \dots, T_{n+1})$, so by Proposition 5.3, so is X. Thus θ defines a section $s \in \mathscr{D}(X)$ with $j_{\#} s = f$. Injectivity is clear. $\square$

Finally, suppose that we have a commutative ring B with 1 containing a Noetherian subring Z with $(\mathscr{T}\text{-spec}\, Z, \mathscr{D}) \cong (\mathscr{T}\text{-spec}\, R, \mathscr{D})$ for some model R of $\mathscr{T}$. We then consider the collection $\{B_i\}_{i \in I}$ of subrings of B which are finitely generated over Z. This forms a filtered inductive system under inclusion with $B \cong \varinjlim B_i$.

Let $A_n = A_n(R)$. Then each $(\mathscr{T}\text{-spec}\, B_i, \mathscr{D}) \cong (\mathscr{T}\text{-spec}\, A_{n_i}/\mathscr{I}_i, \mathscr{D})$ for some $n_i \in \mathbb{N}$ and some finitely generated ideal $\mathscr{I}_i \subset A_{n_i}$. Thus $\mathscr{T}$-spec B_i is LRS-isomorphic to $(X_i, \mathscr{D})$ for some $X_i \in \bar{\mathscr{C}}(\mathscr{T}\text{-spec}\, A_{n_i})$. By the last proposition, this is isomorphic to $(\varrho(X_i), \mathscr{D})$, which is just a closed subset of some affine space together with the sheaf of definable functions. Proposition 6.4 implies

Lemma 6.7. $(\mathscr{T}\text{-spec}\, B, \mathscr{D}) \cong \varprojlim(\varrho(X_i), \mathscr{D})$. $\square$

This is a generalized 'Tarski principle', see Theorem 8.3 below.

7. Continuous definable sections

Suppose $X \in \mathscr{C}(\mathscr{T}\text{-spec}\, A)$, $s \in \mathscr{D}(X)$, and $Y \in \mathscr{C}(\mathscr{T}\text{-spec}\, A[T])$. Then $\Gamma(s) \cap Y$ is constructible, so $\mathrm{pr}(\Gamma(s) \cap Y)$ is constructible by Proposition 1.5. Thus definable sections are continuous in the constructible topologies. In this section we define a subsheaf of $\mathscr{D}$ consisting of sections continuous in the *spectral* topology.

The topological notion of a continuous section s is a section such that $\mathrm{pr}(\Gamma(s) \cap Y)$ is open for all open Y. Using Corollary 2.5, this is equivalent to saying that $\alpha \to \beta \Rightarrow s(\alpha) \to s(\beta)$. But it is well known that the set of such sections do not form a ring (examples for the real spectrum are due to Schwarz and Delfs). The way to circumvent this problem, first used by Delfs [6] for the real spectrum, is to use a 'closed and bounded graph' condition. For this we need a model-theoretic notion of 'bounded'.

Definition 7.1. If $\mathscr{T}$ is any theory in a language $\mathscr{L}$, a formula $\theta(T, U_1, \dots, U_n) \in \mathscr{L}$ is *$\mathscr{T}$-bounded in T* if whenever $f: S \to \mathscr{N}$ is a homomorphism from a substructure of a model $\mathscr{M}$ of $\mathscr{T}$ into another model of $\mathscr{T}$ such that $\mathscr{M} \models \exists T(\theta(T, \bar{s}))$, we can extend f to a substructure $\tilde{S}$ containing S such that $\tilde{S} \models \exists T(\theta(T, \bar{s}))$.

For example, the closed formula $U \neq 0 \wedge TU = 1$ is ACF- and RCF-bounded but not RCVR-bounded because a unit can get mapped to a non-unit. The closed formula $\theta(T, U) =$ '$(U = 0 \wedge T = 0) \vee (TU = 1)$', defining a function f with $f(u) = u^{-1}$ for $u \neq 0$ and $f(0) = 0$, is not bounded in RCF or ACF. To see this, choose a field $\mathscr{F}$, a rank one valuation subring $\mathscr{B}$, and an $m \neq 0$ in the maximal ideal $\mathscr{M}$ of $\mathscr{B}$. $(\mathscr{B}/\mathscr{M}) \models \exists T\theta(T,0)$ with 0 as the only witness. But $\mathscr{F} \models \exists T\theta(T,m)$ with m^{-1} as the only witness, and we cannot extend the projection $\mathscr{B} \to \mathscr{B}/\mathscr{M}$ to include m^{-1}.

Definition 7.2. Let $\mathscr{U}$ be open constructible and $s \in \mathscr{D}(\mathscr{U})$. Let $\psi \in \mathscr{L}_A$ be a sentence defining $\mathscr{U}$ and $\theta(T)$ a $\mathscr{T}_{A[T]}$-formula defining $\Gamma(s)$. We say that s is *closed and bounded* if $\theta(T)$ is $\mathscr{T}_A$-closed and $\psi \wedge \theta(T)$ is $\mathscr{T}_A$-bounded in T.

We can interpret this definition in terms of specializations. Suppose $\alpha, \beta \in \mathscr{U}$ and $\alpha \to \beta$. We let S_α and S_β be the substructures of k_α and k_β generated by $\alpha(A)$ and $\beta(A)$, and let $f: S_\alpha \to S_\beta$ be the $\mathscr{L}$-homomorphism which exists as a result of the specialization. Since $s(\alpha)$ is the unique witness to $\exists T\theta(T)$ in k_α, boundedness implies that we can extend f to $S_\alpha[s(\alpha)]$. Since θ is closed, $k_\beta \models \theta(f(s(\alpha)))$, and using the fact that s is a definable section again, we see $s(\beta) = f(s(\alpha))$. Thus $f(\alpha) \to f(\beta)$. We have shown

Lemma 7.3. *Closed and bounded sections are continuous.* □

On the other hand, suppose $\varphi(T) \in \mathscr{L}_A$. To show $\varphi(T)$ is closed and bounded, we need to check the homomorphism conditions given in Lemma 2.4 and Definition 7.1 for the theory $\mathscr{T}_A$. But an $\mathscr{L}_A$-substructure S of a model of $\mathscr{T}_A$ contains an S_α for some $\alpha \in \mathscr{T}\text{-spec}\, A$. Thus we can show that s is closed and bounded by considering a specialization $\alpha \to \beta$ and an $\mathscr{L}_A$-homomorphism $f: S \to k_\beta$ extending the associated homomorphism $S_\alpha \to S_\beta$. For boundedness we assume $k_\alpha \models \varphi(t)$ and show that f can be extended to $S[t]$. For closure we assume that f has been extended and check that $k_\beta \models \varphi(f(t))$.

Proposition 7.4. *Let $\mathscr{S}(\mathscr{U})$ denote the set of closed and bounded sections over $\mathscr{U}$. Then $\mathscr{S}(\mathscr{U})$ is a commutative ring with 1 and $\mathscr{S}$ is a subsheaf of $\mathscr{D}$.*

Proof. We check that $\mathscr{S}(\mathscr{U})$ is closed under addition. The same method yields closure under other operations and the sheaf axiom for finite covers as well.

Let $s_1, s_2 \in \mathscr{S}(\mathscr{U})$, let ψ be an open sentence defining $\mathscr{U}$, and suppose $\theta_i(T)$ defines s_i. Then $s_1 + s_2$ is defined by the formula

$$\varphi(T) = `\neg\psi \vee \exists UV(\theta_1(U) \wedge \theta_2(V) \wedge T = U + V)'.$$

Suppose $\alpha \to \beta$ in $\mathscr{U}$ and let $f: S \to k_\beta$ a homomorphism as in the previous proposition. The unique witness to $\exists T\varphi(T)$ in k_α is $s_1(\alpha) + s_2(\alpha)$. Since each s_i is bounded, we can extend f to $S[s_1(\alpha), s_2(\alpha)]$. This substructure contains $s_1(\alpha) + s_2(\alpha)$, and clearly $k_\beta \models \varphi(f(s_1(\alpha) + s_2(\alpha)))$. □

The sheaf $\mathscr{S}$ is harder to work with than the sheaf $\mathscr{D}$. It is not clear that Spec $\mathscr{S}(\mathscr{T}\text{-spec}\, A)$ is LRS-isomorphic to $(\mathscr{T}\text{-spec}\, A, \mathscr{S})$, although this often is true.

Certainly $\mathscr{S}$ is not flasque in general. If we are willing to make further restrictions on $\mathscr{T}$ (valid in our examples), then we can at least address the problem of inverting sections whose values are units.

Definition 7.5. A theory $\mathscr{T}$ is *local* if all models of $\mathscr{T}$ are local rings and if $\exists T(UT=1)$ is an open predicate in T.

If $\mathscr{T}$ is local, $s\in\mathscr{S}(\mathscr{U})$ is closed and bounded, and $s(\alpha)\in k_\alpha^\times$ for all $\alpha\in\mathscr{U}$, then $\neg\psi\vee\exists U(\theta(U)\wedge UT=1)$ defines a section $s^{-1}\in\mathscr{D}(\mathscr{U})$. Applying the same argument as in the proof of Proposition 7.4 yields that s^{-1} is closed and bounded, so $s^{-1}\in\mathscr{S}(\mathscr{U})$. It follows that ($\mathscr{T}$-spec $A,\mathscr{S}$) is a locally ringed space.

Proposition 7.6. *The assignment $\mathscr{U}\mapsto\mathscr{S}(\mathscr{U})$ for basic open sets defines a sheaf of commutative rings with* 1 *on $\mathscr{T}$-*spec *A in the spectral topology or in the restricted topology $\mathscr{C}^\circ$ ($\mathscr{T}$-*spec *A). For any $X\subset\mathscr{T}$-*spec *A, $\mathscr{D}(X)$ is an $\mathscr{S}(X)$-module. Furthermore, $\mathscr{T}$-*spec *is a functor* CRO $\rightsquigarrow$ RS *into the category of ringed spaces, and if $\{A_i\}_{i\in I}$ is a filtered inductive system in* CRO, *then $\mathscr{T}$-*spec*$(\varinjlim A_i)\cong\varprojlim\mathscr{T}$-*spec *$A_i$. If $\mathscr{T}$ is a local theory, then ($\mathscr{T}$-*spec *$A,\mathscr{S}$) is a functor into* LRS *and $\mathscr{S}$ is a sheaf of ($\mathscr{T}$-*spec *$A,\mathscr{R}$)-modules, where $\mathscr{R}$ denotes the sheaf of rational functions.*

Proof. Let $f:A\to B$ be a CRO-morphism and s be a closed and bounded section over the open set $\mathscr{U}$. We need to show that $f_{\#}s$ is closed and bounded over $f^{*-1}(\mathscr{U})$. But the $\mathscr{T}_A$-closedness of $\theta(T)$ boundedness of $\psi\wedge\theta(T)$ are statements about all models of $\mathscr{T}_A$ and hence are consequences of $\mathscr{T}$ plus some formula $\sigma\in\Delta^+A$. By Remark 6.5, this gives functoriality and the same remark yields the continuity result for the functor. The rest is clear. □

The discussion at the end of the last section concerning affine space, and in particular Lemma 6.7, is still valid for the sheaf $\mathscr{S}$ if we use the appropriate topology. This helps checking which sections are closed and bounded and whether or not Proposition 6.2 holds for $\mathscr{S}$($\mathscr{T}$-spec A). We go through our examples. All are local theories.

For the theory ACF of algebraically closed fields, closed bounded sections define continuous definable functions on affine varieties. Since the root of a polynomial is never uniquely determined over ACF unless the polynomial is linear, such functions are rational. So $\mathscr{S}=\mathscr{R}$ and we get the usual Zariski spectrum functor. In particular, Proposition 6.2 holds for $\mathscr{S}$.

For RCF a closed bounded section defines a (continuous) semialgebraic function on a closed subvariety of some $\mathbb{A}^n$. Such functions are locally bounded by rational functions [6]. Conversely,

Lemma 7.7. *Let $s\in\mathscr{D}(\mathscr{U})$ be a section over $\mathscr{U}\in\mathscr{C}^\circ$*(RCF-spec *$A$) such that $\Gamma(s)$ is closed and locally bounded by sections in $\mathscr{R}(\mathscr{U})$. Then $s\in\mathscr{S}(\mathscr{U})$.*

Proof. We need to check boundedness. Suppose $\alpha \to \beta$ in $\mathscr{U}$. Our hypothesis means there is an open constructible neighborhood $\mathscr{V}$ of β and elements $a, b \in A$ such that $\gamma(b) \neq 0$ and $|s(\gamma)| < \gamma(a)/\gamma(b)$ for all $\gamma \in \mathscr{V}$. Since $\beta \in \mathscr{V}$ so is α. If $\mathfrak{p} = \alpha(\ker(\beta))$, then any homomorphism $f: S \to k_\beta$ can be extended to the real closure of $S_{(\mathfrak{p}S)}$ in k_α. This includes $s(\alpha)$. $\square$

This gives a concrete description of $\mathscr{S}$. Using elementary analysis to study semialgebraic functions on closed semialgebraic subsets of R^n, one can show that Proposition 6.2 holds for the ring of abstract semialgebraic functions on a closed constructible subset of RCF-spec A_n, and this gives Proposition 6.2 in general.

Lemma 7.8. *Let $s \in \mathscr{D}(\mathscr{U})$ be a section over $\mathscr{U} \in \mathscr{C}^\circ$(RCVR-spec A) such that $\Gamma(s)$ is closed and s is continuous. Then $s \in \mathscr{S}(\mathscr{U})$.*

Proof. We check boundedness. Suppose $\alpha \in A$. Let

$$N = \{a \in A \mid \beta(a) \neq 0 \text{ for all } \beta \in \mathscr{U} \text{ with } \alpha \to \beta\}.$$

Let $\mathscr{O}$ be the convex hull of $\alpha(N)^{-1}\alpha(A)$ in $\mathrm{QF}(k_\alpha)$. Then, referring to the description of specialization given in Section 4, $\mathscr{O}$ is the intersection of all $\mathscr{O}^\beta$ for $\beta \in \mathscr{U}$ with $\alpha \to \beta$. It follows that if $\Gamma(s)$ is closed, then $s(\alpha) \in \mathscr{O}$. But any homomorphism $f: S \to k_\beta$ extends to a ring containing $\mathscr{O}$. $\square$

Let $\mathscr{U} \in \mathscr{C}^\circ$(RCF-spec A) and let $s \in \mathscr{D}(\mathscr{U})$ be continuous and have a closed graph. Using notation from the end of Section 4, we identify $\mathscr{U}$ with $\mathscr{O}(\mathscr{U})$ in RCVR-spec A. This is contained in the open constructible set $\mathscr{U}' = q^{-1}(\mathscr{U})$ (see Proposition 4.6). Suppose $\alpha \to \beta$ in $\mathscr{U}'$. If $s \in \mathscr{S}(\mathscr{U})$, i.e. if s is RCF-bounded, then the associated RCF-specialization map from $F_\alpha = q(\alpha) \to q(\beta) = F_\beta$ extends to include $s(\alpha)$. The same follows for the associated RCVR-specialization map $\alpha \to \beta$. Moreover, the formula for s defines an RCVR section which is in $\mathscr{S}(\mathscr{U}')$. Restrictions from $\mathscr{S}(\mathscr{U}')$ to $\mathscr{U}$ are clearly RCF sections in $\mathscr{S}(\mathscr{U})$. This implies:

Theorem 7.9. *Let $X \subset$ RCF-spec A be constructible and let $s \in \mathscr{D}(X)$ be a closed section. Then s is bounded if and only if it can be extended to an $\tilde{s} \in \mathscr{D}(\tilde{X})$ for some constructible $\tilde{X} \subset$ RCVR-spec A containing X.* $\square$

8. Definable sheaves

The sheaves $\mathscr{R}$, $\mathscr{S}$, and $\mathscr{D}$ all have the property that sections are 'generated' by formulae from $\mathscr{L}$. We formalize this situation.

Let $\mathscr{G}$ be a set of formulae of the form $\theta(T_1, \ldots, T_m, U_1, \ldots, U_n)$ for a fixed $m \in \mathbb{N}$ (but n is not fixed). Let X be a constructible subset of $\mathscr{T}$-spec A for some A. We let $\mathscr{G}(X)$ be the set of formulae $\theta(\bar{T}, \bar{a})$ such that $\theta \in \mathscr{G}$, $a_1, \ldots, a_n \in A$, and such that $\mathscr{T}_A \vdash \psi \Rightarrow \exists!\, T_1, \ldots, T_m\, \theta(T_1, \ldots, T_m, \bar{a})$ for some (and hence all) ψ defining X.

We define an equivalence relation on $\mathscr{G}(X)$ such that $\theta(\bar{T},\bar{a})\sim\gamma(\bar{T},\bar{b})$ if and only if $\mathscr{T}_A\vdash\psi\Rightarrow(\theta\Leftrightarrow\gamma)$ for some ψ defining X. We use $\mathscr{G}(X)$ to denote $\mathscr{G}(X)/\sim$, considering formulae to be identified with the equivalence class they represent.

For every pair $(\theta(\bar{T},\bar{U}),\gamma(\bar{T},\bar{V}))\in\mathscr{G}\times\mathscr{G}$, we assume the formula $\theta(\bar{T},\bar{U})*\gamma(\bar{T},\bar{V}):=\exists\bar{U},\bar{V}(\theta(\bar{E},\bar{U})\wedge\gamma(\bar{F},\bar{V})\wedge\bar{T}_1=E_1*F_1\wedge\cdots\wedge T_m=E_m*F_m)$ is in $\mathscr{G}$, where '$*$' = '$+$' or '$\times$'. We let '0' be the formula '$T_1=\cdots=T_m=0$' and '1' be the formula '$T_1=\cdots=$

More generally, if φ is a k-ary function symbol in our language $\mathscr{L}$, we close $\mathscr{G}$ up with respect to φ as done for '$*$' above. If ϱ is a k-ary predicate in $\mathscr{L}$, and $s_1,\dots,s_k\in\mathscr{G}(X)$ with s_i defined by the formula $\theta_i(T_{1,i},\dots,T_{m,i},\bar{a}_i)$, then we say that $\varrho(x_1,\dots,s_k)$ holds if $\mathscr{T}_A\vDash\psi\Rightarrow(\theta_1\wedge\cdots\wedge\theta_k\Rightarrow\varrho(T_{1,1},\dots,T_{1,k})\wedge\cdots\wedge\varrho(T_{m,1},\dots,T_{m,k}))$. This makes $\mathscr{G}(X)$ into an $\mathscr{L}$-structure.

We also assume that $\mathscr{G}$ is closed under disjunction.

Lemma 8.1. *For any commutative ring A with* 1, *the assignment $X\mapsto\mathscr{G}(X)$ defines a sheaf of rings on $\mathscr{T}$-*spec *A in the constructible* (*and hence in the spectral*) *topology.*

Proof. If $\mathscr{X}(\psi)\subset\mathscr{X}(\delta)$, then $\mathscr{T}_A\vdash\psi\Rightarrow\delta$. This guarantees a presheaf structure. Suppose $\mathscr{T}_A$ proves $\psi\Rightarrow\exists!\,\bar{T}\theta(\bar{T},\bar{a})$, $\delta\Rightarrow\exists!\,\bar{T}\gamma(\bar{T},\bar{b})$, and $\psi\wedge\delta\Rightarrow\theta(\bar{T},\bar{a})\Leftrightarrow\gamma(\bar{T},\bar{b})$. Then $\mathscr{T}_A\vdash(\psi\vee\delta\Rightarrow\exists!\,\bar{T}(\theta(\bar{T},\bar{a}0\vee\gamma(\bar{T},\bar{b}))))$. Using this, one sees that $\mathscr{G}$ is actually a sheaf. □

We call $\mathscr{G}$ a *definable sheaf of functions* on the $\mathscr{G}$-spectrum. The sheaves $\mathscr{R}$, $\mathscr{S}$, and $\mathscr{D}$ are examples.

Note that $\mathscr{G}$ is really generated by formulae of the form $\sigma(\bar{U})\vee\gamma(\bar{U})\vee\theta(T,\bar{U})$ with σ a positive q.f. formula in the language of rings, γ an *open* formula, θ closed, and $\sigma\wedge\gamma\wedge\theta$ bounded in T. If $\mathscr{T}_A\vdash\psi\Rightarrow\exists!\,T\sigma\wedge\gamma\wedge\theta$, the corresponding section is defined over the *open* constructible set $\mathscr{X}(\gamma)$ containing $\mathscr{X}(\psi)$.

We can always use this trick to define our sheaf in the Grothendieck topology $\mathscr{C}^\circ(\mathscr{T}$-spec $A)$. In this case we will say that $\mathscr{G}$ is *defined over open formulae.*

Another example of a definable sheaf of functions is the canonical Nash sheaf on the real spectrum, see [10].

The methods used for $\mathscr{D}$ and $\mathscr{S}$ yield

Proposition 8.2. *The assignment $A\mapsto(\mathscr{T}$-*spec *$A,\mathscr{G})$ defines a functor* CRO $\rightsquigarrow$ RS *which is continuous with respect to filtered direct limits in* CRO. *This is true in the constructible topology and also in the restricted open and spectral topologies if $\mathscr{G}$ is defined over open formulae.* □

The Tarski principle of semialgebraic geometry has a general formulation for definable sheaves of functions. $\mathscr{G}$ defines sheaves of fibred functions on affine space for any model R of $\mathscr{T}$. Suppose a ring B contains a Noetherian ring Z with

$(\mathcal{T}\text{-spec}\, Z, \mathcal{G}) \cong (\mathcal{T}\text{-spec}\, R, \mathcal{G})$ for some model R of $\mathcal{T}$. Then, as in Lemma 6.7, we find a filtered inductive system of ringed spaces $\{(\varrho(X_i), \mathcal{G})\}_{i \in I}$ with each $\varrho(X_i)$ a closed subset of affine space such that $(\mathcal{T}\text{-spec}\, B, \mathcal{G}) \cong \varprojlim(\varrho(X_i), \mathcal{G})$.

Suppose that we are given finitely many constructible sets Y_j in $\mathcal{T}$-spec B and finitely many sections s_{jk} on each of the Y_j. Let P be any statement concerning the $\mathcal{L}$-structure of the $\mathcal{G}(Y_j)$ as applied to the s_{jk} and the (spectral) openness or closedness of the Y_j. We shall call P a *definable statement*. Using Proposition 8.2 we get

Theorem 8.3. *Let B be a commutative ring with* 1 *containing a Noetherian subring Z such that $(\mathcal{T}\text{-spec}\, Z, \mathcal{G})$ is* RS-*isomorphic to $(\mathcal{T}\text{-spec}\, R, \mathcal{G})$ for a model of $\mathcal{T}$ and suppose we are given finitely many $Y_j \in \mathcal{C}(\mathcal{T}\text{-spec}\, B)$, $s_{jk} \in \mathcal{G}(Y_j)$ and a definable statement P concerning the Y_j and s_{jk}. Then there is a closed constructible subset X of some affine space $\mathbb{A}^n(R)$ together with a morphism $(f, f_\#) : (X, \mathcal{G}) \to (\mathcal{T}\text{-spec}\, B, \mathcal{G})$ of spaces with sheaves of rings which are $\mathcal{L}$-structures, such that P is true if and only if the corresponding statement is true for the sets $f^{-1}(Y_j)$ and sections $f_\#(s_{jk})$.* $\square$

This result is (only) useful if one understands the functions $\mathcal{G}$ defines on closed constructible subsets of affine space.

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